Properties of a logistic map with a sectional discontinuity

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We have studied numerically the properties of the logistic map with a single sectional discontinuity at $x=x_d$. We give the main features of these maps over a wide range of x_d, including accumulation points, inverse cascades, bifurcation diagrams, basins of attraction, and a new superimposition rule. We find that the main characteristics of the logistic map with a discontinuity at the origin [T. T. Chia and B. L. Tan, Phys. Rev. A **45**, 8441 (1992)], such as the occurrence of inverse cascades, and the validity of rule I, rule II (which are rules for determining whether higher-level cascades exist), and the summation rule, are still retained in these new discontinuous maps, implying that these properties and rules are universal in discontinuous maps. However, there are important differences as well, such as the number of inverse cascades and the types of routes to chaos which may include a period-doubling route, with period doublings occurring at the same values of a_{PD} for different values of x_d, and an "alternating" route. Further, we find that in the chaotic regions of these maps, either the modified summation rule holds or there exists the following period-doubling sequence: $2\rightarrow4\rightarrow8\rightarrow16\rightarrow32\rightarrow\cdots$, which may also exist in the periodic regions, depending on the values of x_d.

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I. INTRODUCTION

The discontinuous logistic map with a sectional discontinuity at $x=x_d=0$

$$x_{n+1}=\begin{cases}1-a|x_n|^2 & \text{if } x_n>0 ,\\ 0.9-a|x_n|^2 & \text{if } x_n\le 0 ,\end{cases} \tag{1}$$

has been extensively studied recently [1–7]. As the parameter a increases, the orbits of this map are found to be periodic until a critical value a_c is reached when chaos sets in. In the periodic region, there are many different ranges of a, with all the stable cycles in each range having the same period P, which is a function of the range. These constant periods (referred to as *terms*) can be grouped into six different nonoverlapping, first-level inverse cascades such that in each cascade, P decreases in an arithmetic progression as a increases. The terms of the first-level inverse cascades occupy the largest ranges and therefore they are the easiest to be found computationally [1].

In a first-level inverse cascade, it has been found that if an empirical rule, rule I, is obeyed, then between any two consecutive terms, there exist many smaller ranges of the parameter a, over each of which there are stable cycles with identical periods. (See Appendix.) Further, conjecture I implies that new stable cycles will also exist between any other two consecutive terms of this first-level cascade. The periods of all these new cycles are given by an empirical rule known as the summation rule, which involves a process similar to that which generates the terms of the Farey sequence [1]. As a consequence, these new smaller ranges, located between two successive terms of the first-level inverse cascade, form the second-level inverse cascade as well as the second-level direct cascade. Further, the summation rule implies that there will exist higher-level direct and inverse cascades embedded between successive terms of a cascade one level lower.

On the other hand, if rule I does not hold between any two consecutive terms of the first-level inverse cascade, then rule II holds, which implies that no new cycles will exist between these terms.

Of the six first-level inverse cascades present, no new cycles can be found between any two consecutive terms of the first two cascades as rule II is obeyed, and consequently conjecture II should hold. However, in each of the remaining four cascades, new cycles exist between any two consecutive terms as rule I, and consequently conjecture I, are satisfied. In the chaotic region of the map, there exist windows of stable orbits over some of which they satisfy a modified form of the summation rule.

The above map is the only one found so far in which there exist both direct and inverse cascades with either rule II or rule I and the accompanying summation rule holding. We might wonder if these are universal properties of general discontinuous maps. Hence, the purpose of this paper is to examine other one-dimensional discontinuous maps to ascertain whether these properties still hold. In particular, we shall study the behavior of the one-dimensional discontinuous logistic map with a sectional discontinuity at different values of x_d, i.e., the following map

$$F(x_n)=\begin{cases}1-a|x_n|^2 & \text{if } x_n>x_d ,\\ 0.9-a|x_n|^2 & \text{if } x_n\le x_d ,\end{cases} \tag{2}$$

where x_d varies from -0.3 to 0.8. Though this map differs from that of Eq. (1) by only the location of the sectional discontinuity, we shall see below that some of its properties may be quite different.

TABLE I. First-level inverse cascades and other sequences in the periodic region at various values of x_d. The symbol PD denotes period doubling. Chaos occurs immediately after the last cycles shown for all the values of x_d except those labeled by (*), for which there exist other terms that appear to belong to higher-level inverse and direct cascades before the onset of chaos.

x_d	Sequences
−0.3(*)	$1\xrightarrow{\text{PD}}2\xrightarrow{\text{PD}}4, \cdots \to 50\to 46\to 42\to 38\to 34$
−0.2(*)	$1\xrightarrow{\text{PD}}2,4\xrightarrow{\text{PD}}8, \cdots \to 60\to 52\to 44\to 36\to 28$
−0.1	$1\xrightarrow{\text{PD}}2,6,4\xrightarrow{\text{PD}}8,14\xrightarrow{\text{PD}}28,44$
−0.01	$1\xrightarrow{\text{PD}}2,8,6,4\xrightarrow{\text{PD}}8,2\xrightarrow{\text{PD}}4, \cdots \to 43\to 39\to 35\to 31\to 27,31,35,39,43$
0.01	$1\xrightarrow{\text{PD}}2, \cdots \to 14\to 12\to 10\to 8\to 6,6,4\xrightarrow{\text{PD}}8,2\xrightarrow{\text{PD}}4$
0.1	$1\xrightarrow{\text{PD}}2, \cdots \to 12\to 10\to 8\to 6\to 4,4,2\xrightarrow{\text{PD}}4\xrightarrow{\text{PD}}8,37\to 103\to 37\to 103\to 37\to \cdots$
0.2	$1\xrightarrow{\text{PD}}2, \cdots \to 10\to 8\to 6\to 4\to 2\xrightarrow{\text{PD}}4\xrightarrow{\text{PD}}8\xrightarrow{\text{PD}}16\xrightarrow{\text{PD}}32\xrightarrow{\text{PD}} \cdots$
0.3	$1\xrightarrow{\text{PD}}2, \cdots \to 10\to 8\to 6\to 4\to 2\xrightarrow{\text{PD}}4\xrightarrow{\text{PD}}8\xrightarrow{\text{PD}}16\xrightarrow{\text{PD}}32\xrightarrow{\text{PD}} \cdots$
0.4(*)	$1\xrightarrow{\text{PD}}2, \cdots \to 20\to 18\to 16\to 14\to 12$
0.5	$1\xrightarrow{\text{PD}}2, \cdots \to 98\to 96\to 94\to 92\to 90$
0.6	$1\xrightarrow{\text{PD}}2, \cdots \to 175\to 173\to 171\to 169\to 167$
0.65	$1\xrightarrow{\text{PD}}2, \cdots \to 57\to 55\to 53\to 51\to 49$
0.7	$1\xrightarrow{\text{PD}}2, \cdots \to 392\to 390\to 388\to 386\to 384$
0.8(*)	$1\xrightarrow{\text{PD}}2, \cdots \to 12\to 10\to 8\to 6\to 4$

II. FIRST-LEVEL INVERSE CASCADES

The map F given by Eq. (2) for different values of x_d ranging from -0.3 to 0.8 has been studied computationally over the complete range of the parameter a. We have adopted an initial value $x_0=0.5$ in all the computations except in Sec. VI. In general, the behavior is similar to that of Eq. (1), such as the existence of inverse cascade(s) in the periodic region. However, there are major differences as well. Our results are summarized in Tables I and II.

TABLE II. Accumulation points of the first-level inverse cascades given in Table I.

x_d	Accumulation point
−0.3	1.300 67
−0.2	1.288 97
−0.01	1.530 55
0.01	0.990 20
0.1	0.916 74
0.2	0.857 87
0.3	0.815 25
0.4	0.784 65
0.5	0.763 94
0.6	0.752 37
0.65	0.835 39
0.7	0.842 25
0.8	0.879 34

When the discontinuity x_d of F lies between 0 and 0.004, we find that there exist six first-level inverse cascades in the periodic region, just as in the case when $x_d=0$ [1]. Moreover, these six inverse cascades are identical to those of the map with $x_d=0$. These results are not unexpected, as the value of x_d is very small and therefore these maps should have properties similar to those of the map given by Eq. (1).

However, for most other values of x_d, there exists only *one* first-level inverse cascade, while for some other values, *no* such cascade exists. In particular, we have found that when $x_d=-0.1$, no inverse cascade exists, while when x_d takes on any one of the remaining values shown in Table I, only *one* first-level inverse cascade exists. Hence, the number of inverse cascades in discontinuous maps is not a universal property.

Table I gives the period-doubling sequences, the first-level inverse cascades (if they exist), and some other cycles in the *periodic* region for each of the discontinuous maps with the above-mentioned values of x_d. From this table, we conclude that each inverse cascade, which is an arithmetic progression, has a common difference equal to the period immediately below its accumulation point. As this same rule is also observed in the case with $x_d=0$ [1], this rule appears to be universal for general discontinuous maps.

Table II gives the accumulation points a_{acc} of the first-level inverse cascades shown in Table I. At all accumulation points, the true periods are actually infinite, though in practice, due to the finite word length of computers, finite periods that are precision-dependent are found instead [2].

III. RULE I, RULE II, SUMMATION RULE, AND IMPLICATIONS

For all the first-level inverse cascades shown in Table I, except those corresponding to x_d equal to -0.01, 0.6, and 0.65, we find that, between any two consecutive terms of these inverse cascades, rule I holds and consequently conjecture I should also hold, implying that new cycles should exist between any two consecutive terms of these inverse cascades [1]. We are able to confirm that between any two consecutive terms of these first-level inverse cascades, there exist new cycles belonging to higher-level inverse and direct cascades. Further, the periods of these new cycles found are in agreement with those of the predicted periods obtained from the summation rule [1]. Thus, for these inverse cascades, rule I, conjecture I, and the summation rule are all applicable.

For the first-level inverse cascades belonging to the other three maps, namely those with x_d equal to -0.01, 0.6, and 0.65, we find that rule II and conjecture II hold instead, implying that no new cycles should exist between any of their two consecutive terms and that there should be no higher-level inverse or direct cascades. We are able to confirm that new cycles do not exist between any two consecutive terms of these three maps.

Thus we have found that either rule I, which predicts the existence of new cycles between any two consecutive terms of the first-level inverse cascade, and the summation rule, which gives the periods of the new cycles, or rule II, which predicts the nonexistence of new cycles, are also applicable for the map F. We suspect that these three rules are universal for discontinuous maps.

A. Numerical illustrations

Table III illustrates the situation where rule I is satisfied for the cycle with period $P=36$ belonging to the first-level inverse cascade of the map F with $x_d=-0.2$. Here $a=1.2907$, $n_c=31$, and we have used $n=31$ with the notations of Ref. [1].

An example of a case when rule II is satisfied is shown in Table IV, which gives some of the iterates of the $P=209$ cycle belonging to the inverse cascade of F with $x_d=0.6$ when $a=0.7542$. By choosing $n_d=1$ and $n=100$, it is readily verified that rule II holds.

TABLE III. Example illustrating rule I when $x_d=-0.2$ with $n_c=31$, $n=31$, $P=36$, and $a=1.2907$.

Iteration number	Value of iterate
31	0.966 297 583 586 316 459 961 930 070 918 61
67	0.966 323 717 462 255 332 874 158 418 721 90
103	0.966 343 464 729 670 456 964 959 499 019 90
139	0.966 358 423 026 775 179 831 175 982 904 84
175	0.966 369 774 907 440 662 757 659 743 273 05
211	0.966 378 402 064 967 629 613 129 101 453 57
247	0.966 384 965 541 899 762 699 203 440 191 64

TABLE IV. Example illustrating rule II when $x_d=0.6$ with $n_d=1$, $n=100$, $P=209$, and $a=0.7542$.

Iteration	Value of iterate
100	0.604 343 364 502 154 479 308 648 404 144 64
309	0.604 230 339 808 563 325 371 808 550 862 49
518	0.604 329 242 582 485 842 752 295 335 069 41
727	0.604 242 687 015 679 863 301 714 167 409 07
936	0.604 318 428 988 318 241 957 953 547 013 95
1145	0.604 252 143 635 144 037 798 364 091 879 46
1354	0.604 310 148 461 153 292 643 754 446 183 45

As an illustration of the summation rule, consider the first-level inverse cascade

$$\cdots \rightarrow 12 \rightarrow 10 \rightarrow 8 \rightarrow 6 \rightarrow 4$$

belonging to F with $x_d=0.1$. Here rule I is applicable to either one of the consecutive terms 6 and 4, implying the

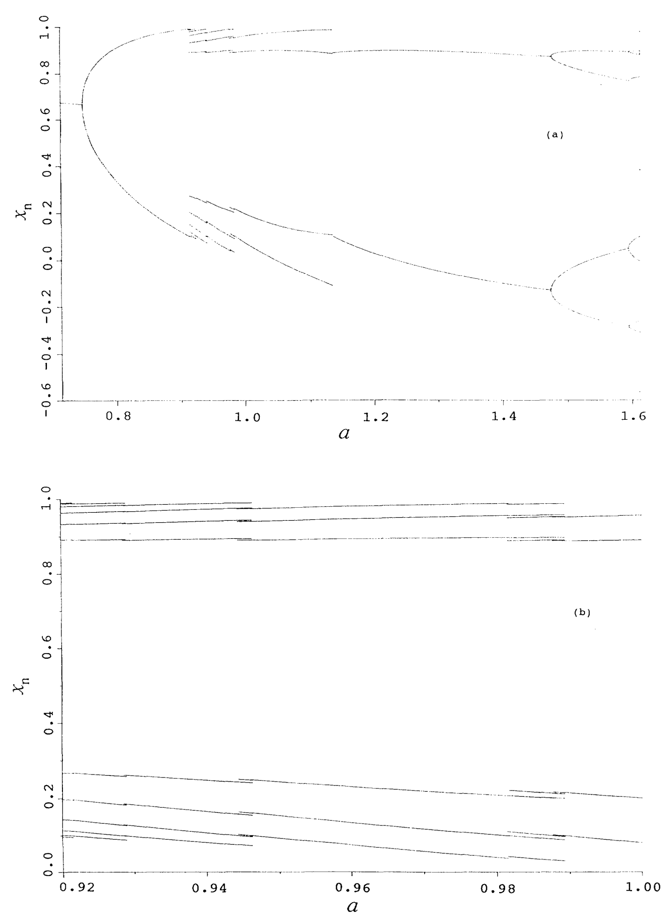

FIG. 1. $x_d=0.1$: (a) Bifurcation diagram in the periodic region of the map. (b) Enlargement of a portion of the first-level inverse cascade shown in (a), where now the occurrence of second-level direct and inverse cascades within the first-level inverse cascade is more visible.

existence of new cycles between these two terms. The summation rule predicts the periods of the new cycles. One new cycle will have a period given by the sum of 6 and 4. By applying the same process to this new cycle with period 10 and either one of the old cycles with period 6 or 4, we can predict the periods of the new cycles lying between them, namely, 16 and 14. Applying the same process once again would lead to the prediction of the following new cycles with periods given by

$$6 \rightarrow \cdots \rightarrow 22 \rightarrow 16 \rightarrow 10 \rightarrow 14 \rightarrow 18 \rightarrow \cdots \rightarrow 4 .$$

From these cycles, or terms, we can extract a second-level inverse cascade

$$\cdots \rightarrow 22 \rightarrow 16 \rightarrow 10 \rightarrow 4 ,$$

as well as a second-level direct cascade

$$6 \rightarrow 10 \rightarrow 14 \rightarrow 18 \rightarrow \cdots$$

with common differences of 6 and 4, respectively. If we continue to use the summation rule, say between 16 and 10 of the second-level inverse cascade, we expect to obtain the following new terms

$$16 \rightarrow \cdots \rightarrow 58 \rightarrow 42 \rightarrow 26 \rightarrow 36 \rightarrow 46 \rightarrow \cdots \rightarrow 10 ,$$

from which we can again extract a third-level inverse cascade

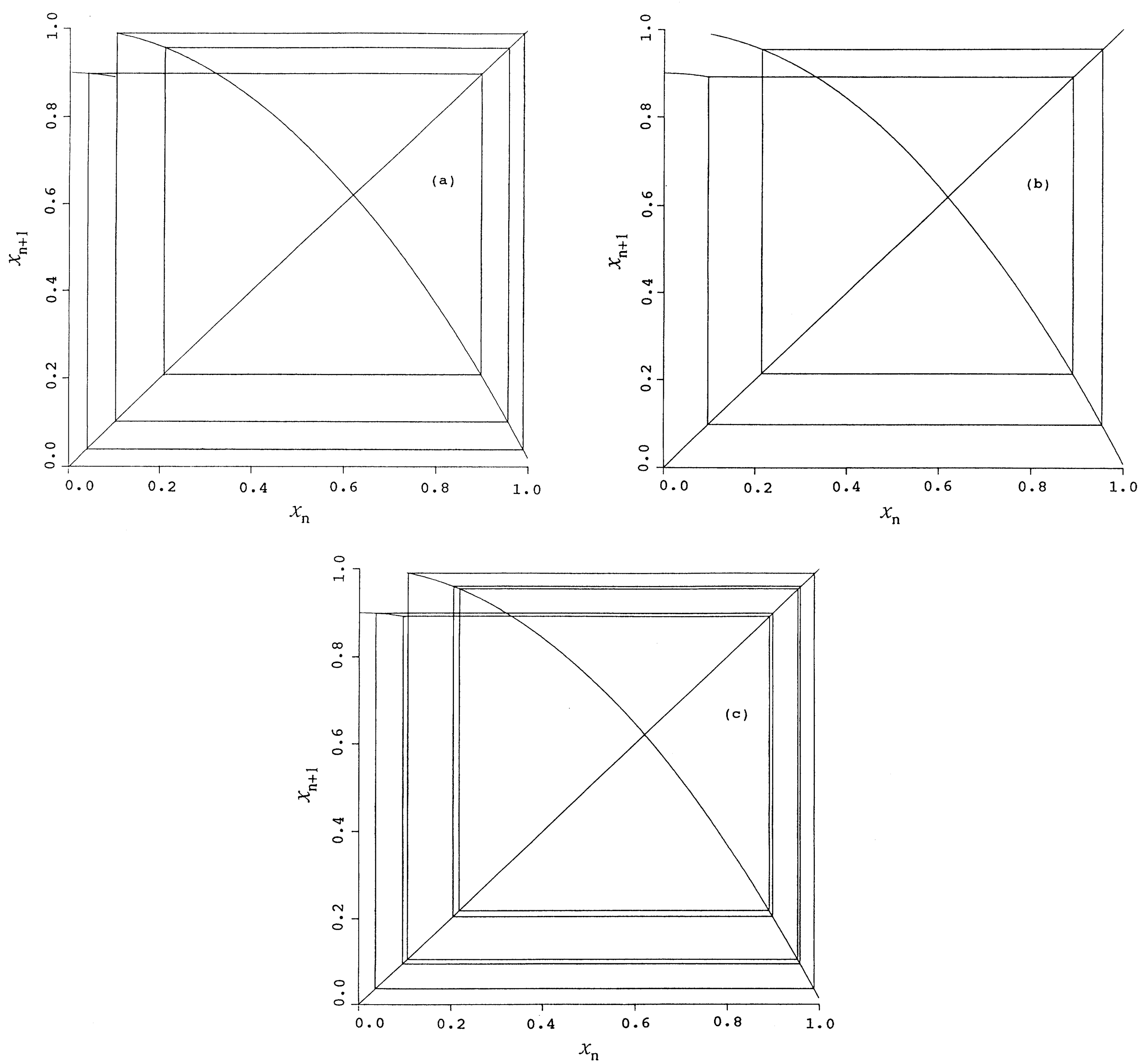

FIG. 2. $x_d = 0.1$: (a) Period 6 cycle at $a = 0.981$. (b) Period 4 cycle at $a = 0.99$. (c) Period 10 cycle at $a = 0.985$, due to the summation rule.

$$\cdots \rightarrow 58 \rightarrow 42 \rightarrow 26 \rightarrow 10 ,$$

as well as a third-level direct cascade

$$16 \rightarrow 26 \rightarrow 36 \rightarrow 46 \rightarrow \cdots ,$$

with common differences of 16 and 10, respectively. The process can be repeated to yield an infinite number of levels of cascades.

We have been able to confirm the existence of all these new cycles with the predicted periods given above.

B. Bifurcation diagrams

An example of a bifurcation diagram for a map F is given in Fig. 1(a). Here $x_d = 0.1$ and only the periodic region of the map is shown. From this figure, we can see the birth of the first-level inverse cascade at the end of the period 2 cycles at a parameter value of 0.916 74, and beyond $a = 1.615 79$, chaos first occurs. Notice that higher terms of the inverse cascade tend to occupy smaller intervals, a behavior that is also observed for Eq. (1) [1].

Figure 1(b) is the enlargement of a region in Fig. 1(a), showing second-level direct and inverse cascades embedded within terms of the first-level inverse cascade. Higher terms of the second-level and higher-level cascades are not visible from the graph, as they occupy narrower intervals.

C. Rule for approximating values of cycle elements in higher-level cascades

While the summation rule allows us to accurately predict the periods of new cycles between any two consecutive terms of a first-level cascade provided rule I is obeyed, by itself it cannot predict the values of the new cycle elements [1]. We find that the values of the new cycle elements belonging to higher-level cascades generated by the summation rule are very simply related to those of the cycle elements belonging to the two consecutive terms. In fact, we can predict the approximate values of the new cycle elements from those of the cycle elements belonging to these two consecutive terms by using the following *superimposition rule:* If x_i $(i = 1,2,3,\ldots,N)$ are the values of the cycle elements of an N-cycle belonging to the end of a term of a first-level inverse cascade and x_j' $(j = 1,2,3,\ldots,N')$ are the values of those of an N'-cycle belonging to the beginning of a consecutive term, such that $N > N'$, then each of the N'-cycle elements x_j' will be close to one of the N-cycle elements x_i. The new cycle with period $N+N'$ will then contain $N+N'$-cycle elements x_k'' $(k = 1,2,3,\ldots,N+N')$ such that every one of these new elements will be close to either an element x_i or to an element x_j'. The values of these elements, x_k'', of this new cycle are then approximately equal to those of the set obtained by superimposing the sets x_i and x_j'.

This rule can be illustrated by using the example provided for the summation rule in Sec. III A. In particular, we shall consider the consecutive terms 6 and 4 belonging to the first-level inverse cascade. Here $N = 6$ and $N' = 4$.

Figure 2(a) shows the map with $x_d = 0.1$ at a parameter $a = 0.981$, which is near the end of the 6-cycle. In Fig. 2(b), we see a 4-cycle at $a = 0.99$, shortly after it is born. Notice that the 4-cycle elements almost coincide with four of the 6-cycle elements. The summation rule predicts a 10-cycle between these two cycles, and this is shown in Fig. 2(c) with $a = 0.985$. If we examine these figures carefully, we can think of the approximate location of the 10-cycle elements along the x axis as being those of the 6-cycle superimposing on top of the 4-cycle. Table V further illustrates this rule for the 10-cycle. This rule can also be used between the 10-cycle and either the 6-cycle or the 4-cycle to give the approximate values of the cycle elements of the 16-cycle and 14-cycle, and so on. These are also illustrated in the same table for all the terms generated by the summation rule from the two consecutive terms 6 and 4:

$$6 \rightarrow \cdots \rightarrow 22 \rightarrow 16 \rightarrow 10 \rightarrow 14 \rightarrow 18 \rightarrow \cdots \rightarrow 4 .$$

We have also found this rule to be applicable to any two consecutive terms of any inverse cascade with any value of x_d when the summation rule holds. Thus we can conclude that the cycle elements of higher-level cascades tend to crowd around the positions of the cycle elements in the first-level inverse cascade along the x axis. Further, we can predict exactly how many of the cycle elements of the higher-level cascades are close to each cycle element of the first-level cascade, as the positions of the new cycle elements are given approximately by the super-

TABLE V. Example illustrating the location of cycle elements in a higher-level cascade with respect to those in the first-level cascade. Periods 6 and 4 are consecutive terms of the first-level cascade when $x_d = 0.1$.

Period of orbit	Number of cycle elements of orbit close to x_i, The cycle elements of orbit with period 6					
	x_1	x_2	x_3	x_4	x_5	x_6
6	1	1	1	1	1	1
22	3	4	4	4	4	3
16	2	3	3	3	3	2
10	1	2	2	2	2	1
14	1	3	3	3	3	1
18	1	4	4	4	4	1
4	0	1	1	1	1	0

imposition of those of the cycle elements of the two consecutive terms.

IV. ROUTES TO CHAOS

Based on the variation of the Lyapunov exponent as a function of the parameter a and the ϵ-a phase diagram for the map given by Eq. (1), where ϵ determines the size of the discontinuity along the y axis, de Sousa Vieira and co-workers believe that chaos first occurs at the "accumulation point of the accumulation points" [3–6]. However, by using our definition of the inverse cascade given earlier [1], we find that the accumulation points do not accumulate at the value of $a=a_c$ where chaos first occurs. Our reasons are as follows. Though this map only possesses six first-level inverse cascades, there exist a possibly infinite number of higher-level inverse and direct cascades, each with an accumulation point, whenever rule I holds [1]. In view of the fact that within each first-level inverse cascade, the range of each term increases with a (see Fig. 1 of Ref. [1]), the separation between each set of accumulation points would tend to get larger towards the beginning of each first-level inverse cascade, i.e., with increasing a. Consequently, if these sets of accumulation points were to accumulate, then they can only do so at the accumulation point of each first-level inverse cascade, which lies at its end and not at the beginning. Thus they cannot accumulate at the point where chaos first arises, which we have found to lie beyond these first-level inverse cascades. There does exist another type of "inverse cascade," with one part lying inside the periodic region and the other part in the chaotic region. However, it is most unlikely that any of each set of possibly densely-packed, higher-level accumulation points of this seventh "first-level inverse cascade" could accumulate at a_c.

For each of the maps F with $x_d=-0.01$, 0.6, and 0.65, we have seen from Sec. III that no new cycles can exist between any two consecutive terms of the first-level inverse cascade, as rule II and conjecture II are satisfied. Hence, in each case, higher-level cascades cannot exist, implying that there is only one accumulation point at the end of the first-level inverse cascade. Therefore, for these three maps, chaos obviously cannot occur by the route known as "accumulation point of the accumulation points."

From Table I, we observe that at the end of the first-level inverse cascades of both maps F with $x_d=0.2$ and 0.3, there exists a series of period doublings until chaos is reached. Thus the route to chaos for these two maps is by *period doubling*, similar to that for the standard logistic map. We note further that the parameters at which period doublings occur, $a_{\rm PD}$, are the same for both these maps and also for maps with values of x_d lying between 0.2 and 0.3. However, the values of $a_{\rm PD}$ are different from those of the logistic map. This is not a surprising result, as in the discontinuous maps, the cycle elements lie on both portions of the maps.

In the case of the map F with $x_d=0.1$, the route to chaos is quite different. By comparing the sequences shown in Table I for $x_d=0.2$ and 0.3 with that for $x_d=0.1$, we see that period doubling does not carry on infinitely when $x_d=0.1$, but instead stops at period-8. As a increases beyond this point, the periods of the stable cycles alternate between 37 and 103 in a cyclic sequence

$$37\rightarrow 103\rightarrow 37\rightarrow 103\rightarrow 37\rightarrow 103\rightarrow \cdots$$

just before the onset of chaos. There exist many (possibly infinite) occurrences of the terms 37 and 103, where the length of parameter occupied by each term of the sequence does not seem to follow any regular pattern. After the onset of chaos, we observe another cyclic sequence

$$21\rightarrow 80\rightarrow 21\rightarrow 80\rightarrow 21\rightarrow 80\rightarrow \cdots .$$

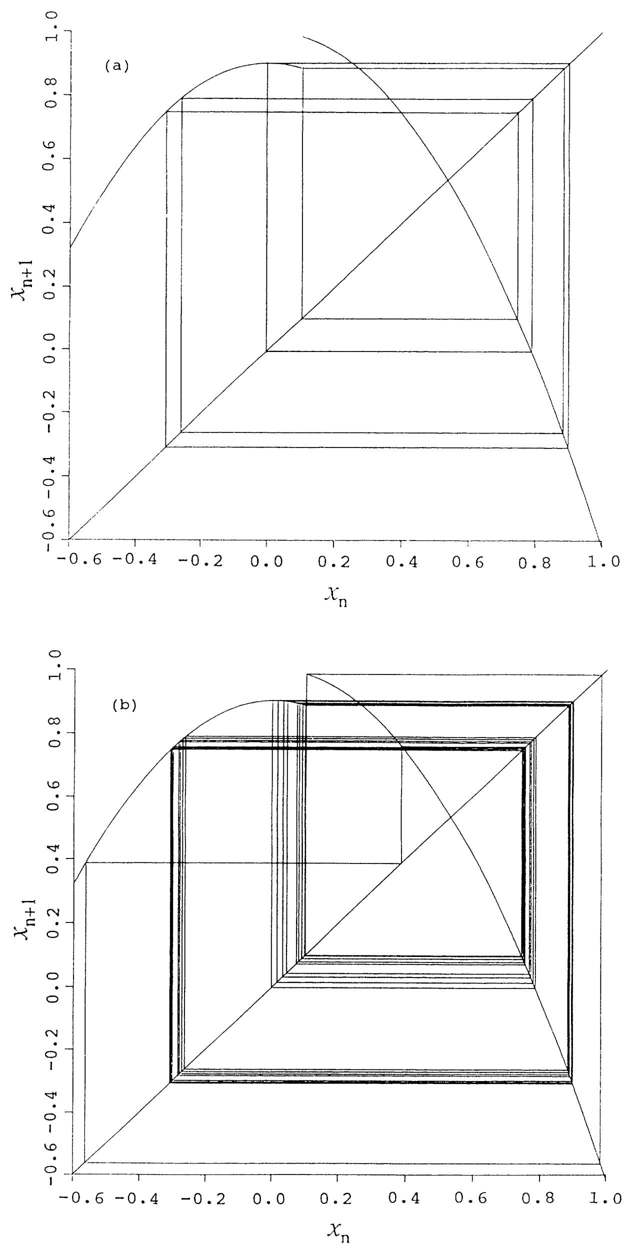

FIG. 3. $x_d=0.1$: (a) Period 8 cycle at $a=1.615\,23$. (b) Period 37 cycle at $a=1.615\,24$. At least one cycle element is close to x_d in both cases.

Hence, the route to chaos for this map is quite different from the period-doubling route. We shall tentatively label this route as the alternating route, the nature of which is still under exploration.

Note that, for this particular map, the route to chaos is not always by the alternating route, as it is basin-dependent. For example, with x_0 equal to 0, we get exactly the same pattern of cycles up to the 8-cycle, i.e., the inverse cascade followed by the period doubling of the 2-cycle up to the 8-cycle, but beyond this the cyclic sequence is replaced by just the two terms 37 and 103, at the end of which chaos occurs.

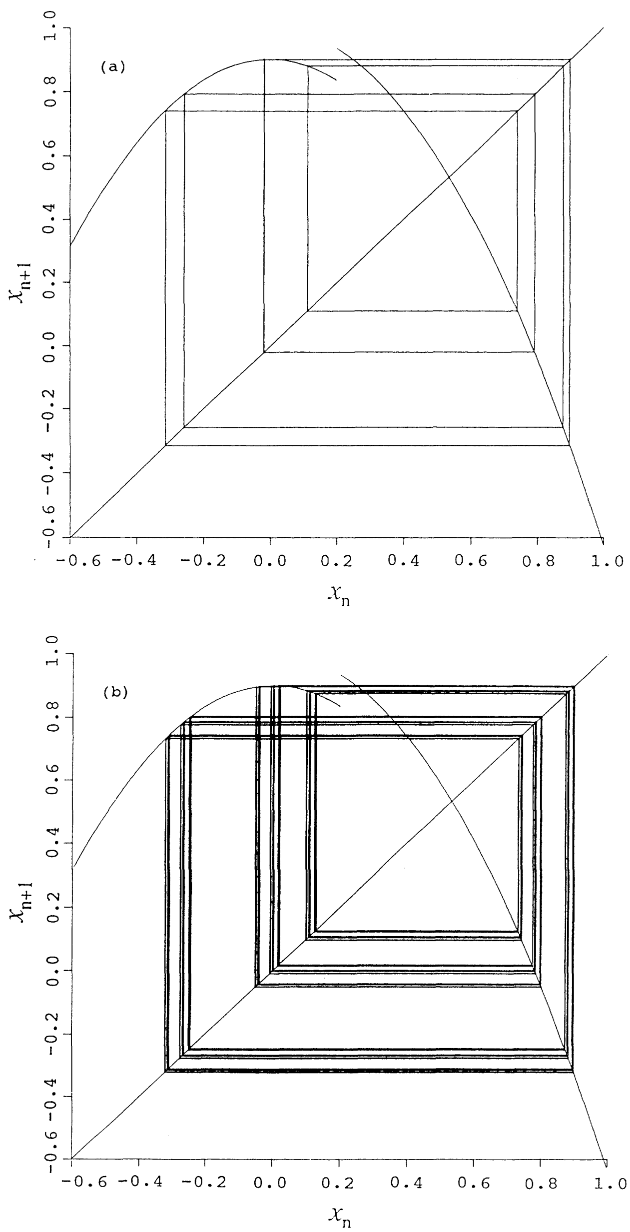

FIG. 4. $x_d = 0.2$: (a) Period 8 cycle at $a = 1.624\,64$, just before period doubling takes place. (b) Period 256 cycle at $a = 1.631\,92$, after several period doublings of the 8-cycle. None of the cycle elements is close to x_d in both cases, unlike the situations in Fig. 3.

That the character of the bifurcation involved in this alternating route to chaos is quite different from that involved in the period-doubling route can be seen graphically as follows. Figure 3(a) shows a period 8 cycle at $a = 1.615\,23$ for the alternating route map, i.e., for the map with $x_d = 0.1$ with one of the cycle elements lying close to the discontinuity of the map. When the parameter a is increased slightly to $a = 1.615\,24$, as shown in Fig. 3(b), the period of the stable cycle is 37 instead, i.e., the 8-cycle has bifurcated into a 37-cycle. Thus this bifurcation occurs when a cycle element is close to the discontinuity. In the case of the period-doubling route, the situation is quite different. The map with $x_d = 0.2$ is shown in Fig. 4(a), which illustrates a period 8 cycle at $a = 1.624\,64$, shortly before it loses its stability and its period doubles into a 16-cycle. In this case, none of the cycle elements is close to the discontinuity, but instead bifurcation occurs because the tangent of F^n [where in this notation, $F^2(x)$ means $F(F(x))$] is equal to -1 at each of these n-cycle elements. This result remains valid even after many period doublings, an illustration of which is given in Fig. 4(b) for the map with $x_d = 0.2$, where the stable cycle has a period $256 = 2^8$ at $a = 1.631\,92$.

The mechanism of the route to chaos for the discontinuous map with other values of x_d is still mysterious. In the case when $x_d = -0.1$, chaos appears to arise in the midst of an increasing sequence, as terms such as $60, 76, 92, \ldots$ can be found in the chaotic region. Similarly, when $x_d = -0.01$, the same behavior is observed, and in this case, we can find the terms 47 and 51 after the onset of chaos. In the case when $x_d = 0.01$, there is no evidence of the presence of any such sequence when chaos arises. For all the maps with values of x_d given in Table I, except those labeled with an asterisk (*), chaos first occurs immediately after the last cycles shown. However, for those labeled with the asterisk (*), beyond the inverse cascades shown, there exist other terms that appear to belong to higher-level inverse and direct cascades before chaos first occurs.

V. MODIFIED SUMMATION RULE AND PERIOD DOUBLINGS IN CHAOTIC REGION

For the map F given by Eq. (2) with $x_d = -0.3$, -0.2, -0.01, -0.1, 0.01, 0.1, or 0.8, the modified summation rule seems to hold in the chaotic region. For instance, between the terms 168 and 68 when $x_d = -0.1$, we find the sequence

$$168 \to \cdots \to 572 \to 404 \to 236 \to 304 \to 372 \to \cdots \to 68$$

interposed with chaos and other periodic windows. However, the following sequence $P \to 4P \to 3P \to 2P \to 3P \to 4P \to P$, which can be widely found in the chaotic region of the map given by Eq. (1) and which satisfies the modified summation rule, cannot be found in the maps F with these values of x_d.

However, for the maps with $x_d = 0.2$, 0.3, 0.4, 0.5, 0.6, 0.65, or 0.7, the periodic cycles in the chaotic region do not seem to obey the modified summation rule. Instead, for these maps, we observe the following period-doubling sequences $2 \to 4 \to 8 \to 16 \to 32 \to \cdots$ either in the periodic or in the chaotic regions. Further, the values of the parameter a when period doubling occurs, a_{PD}, are the same for each of these maps. As observed in Sec. IV, when $x_d = 0.2$ or 0.3, the above *period-doubling* sequence leads to the onset of chaos. However, when $x_d = 0.4$, 0.5, 0.6, 0.65, or 0.7, this period-doubling sequence occurs in the chaotic region, with the 2-cycle being the first periodic cycle after the onset of chaos.

That the bifurcation points for the period-doubling sequence $2 \to 4 \to 8 \to 16 \to 32 \to \cdots$ are identical for each of these maps with values of x_d between 0.2 and 0.7 can be understood graphically. Figure 5(a) illustrates a 32-cycle at $a = 1.6315$ when $x_d = 0.2$, while Fig. 5(b) shows the same cycle at the same value of a but with $x_d = 0.7$. As the cycle elements of both these maps lie outside the region x between 0.2 and 0.7 and these maps have the same shape outside this region, it follows that both sets of cycle elements are identical. Further, it follows that any of these maps with x_d lying between 0.2 and 0.7 and $a = 1.6315$ will possess the same set of cycle elements. Hence, if all the cycle elements of these maps remain outside this region as a varies, period doublings must occur at the same values of a_{PD} for each of these maps.

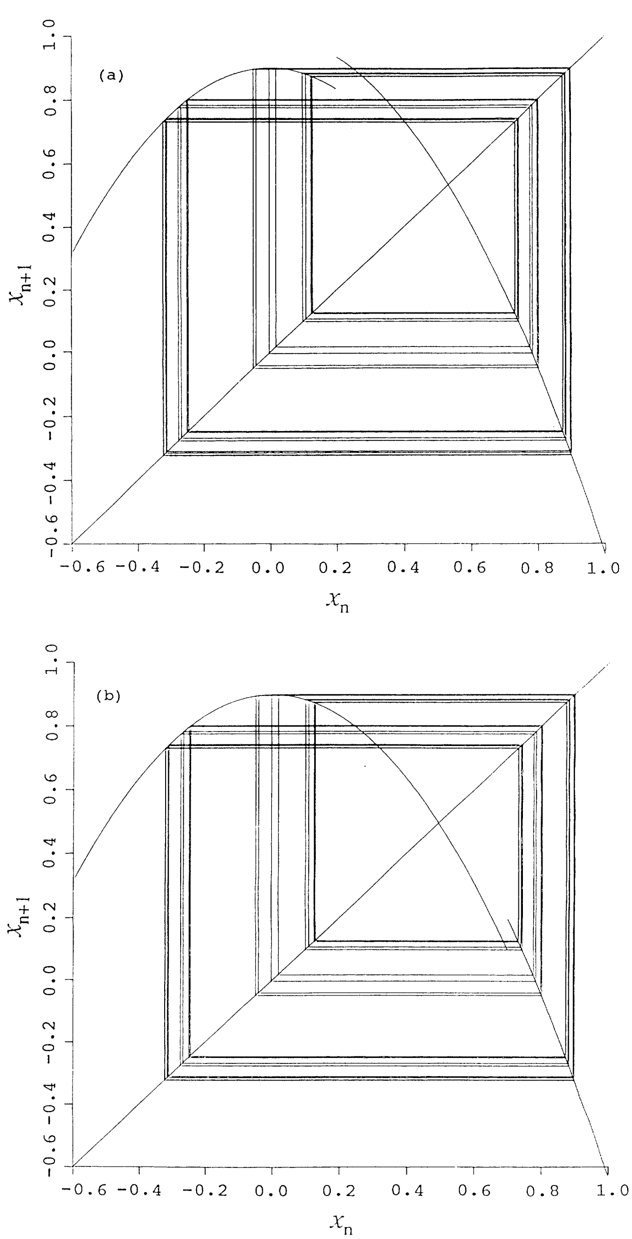

FIG. 5. 32-cycle at $a = 1.6315$: (a) $x_d = 0.2$, (b) $x_d = 0.7$. The two sets of cycle elements are identical for both maps.

We are unable to find such a period-doubling sequence at values of x_d where the modified summation rule seems to hold in the periodic or chaotic region. Further, when such a period-doubling sequence exists in the chaotic region, the modified summation rule does not apply. Hence, it seems that the modified summation rule and this particular period-doubling sequence are mutually exclusive. However, it is possible for other types of period-doubling sequences to coexist with the modified summation rule.

VI. BASINS OF ATTRACTION

In the logistic map, there exists only one basin of attraction, i.e., the eventual orbit is independent of the initial value x_0. However, for the discontinuous logistic map, we find that the number of basins may be either one or two, depending on the value of a.

Let us first illustrate the situation for the bifurcation of the 2-cycle into a 4-cycle for the map F with $x_d = -0.2$. When x_0 is equal to 0.5, which is the initial value we have employed throughout our computations in the other sections of this paper, we observe a 2-cycle when $a = 1.2421$, slightly before the bifurcation of the 2-cycle into a 4-cycle. However, when a different value of x_0 is chosen for the same parameter value, we may get either a 2-cycle or a 4-cycle. This result is summarized in Fig. 6(a), which shows the basins of attraction when $a = 1.2421$. In this figure, for a given value of x_0, a dark vertical line indicates a 2-cycle, while a white vertical line indicates a 4-

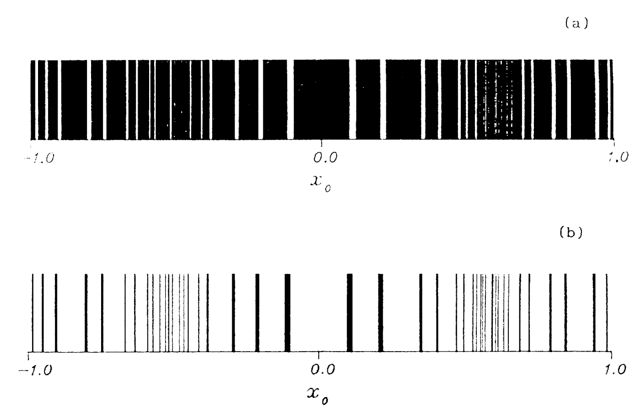

FIG. 6. $x_d = -0.2$: (a) Basins of attraction for the 2- and 4-cycles at $a = 1.2421$, where the shaded regions represent 2-cycles and unshaded regions 4-cycles. (b) Complement of Fig. 6(a), where now the shaded regions represent 4-cycles and unshaded regions 2-cycles.

cycle. Figure 6(b) illustrates the complement of Fig. 6(a), where now a 4-cycle is denoted by a dark vertical line, while a 2-cycle is denoted by a white line. Thus, for some values of x_0, bifurcation from the 2-cycle into the 4-cycle can occur before $a = 1.2421$.

Figure 7(a) further illustrates the two basins over a finite interval of the parameter a. In this diagram, a dark dot denotes a 2-cycle, while a white dot indicates a 4-cycle. When a is small, the periods of the orbits are mostly 2, so that the total area occupied by the white regions in the diagram is tiny. These white regions grow bigger with increasing a. This is intuitively correct, as we expect more 4-cycles to be present at higher values of the parameter.

Another characteristic that is not obvious in Fig. 7(a) is the presence of continuous horizontal white lines over certain ranges of x_0 for large values of a. These can be seen in Fig. 7(b), which is an enlargement of a small region in Fig. 7(a). Here, over a large region in the x_0-a plane, the cycles have periods 4, while over other regions, the cycles have alternating periods of 2 and 4. Thus, if x_0 is chosen to lie in the "alternating" domain, say 0.13, we would observe the cyclic sequence $2 \to 4 \to 2 \to 4 \to 2 \to 4 \to \cdots$, which is illustrated in Fig. 7(b) by the alternating occurrences of dark and white dots.

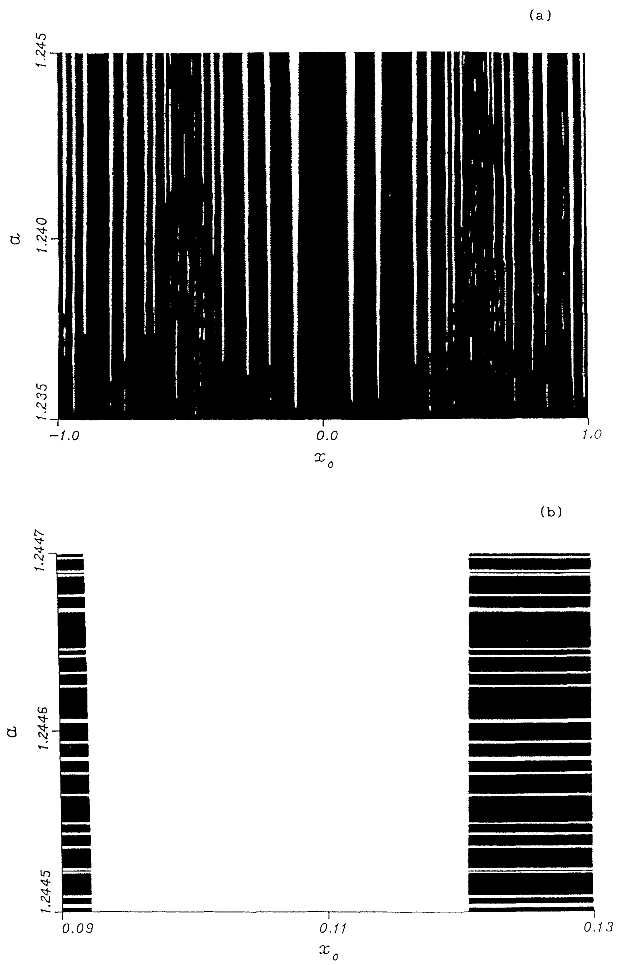

FIG. 7. $x_d = -0.2$: (a) Basins of attraction for the 2- and 4-cycles over an interval of a, where the shaded regions represent 2-cycles and unshaded regions 4-cycles. (b) Enlargement of a small region in (a).

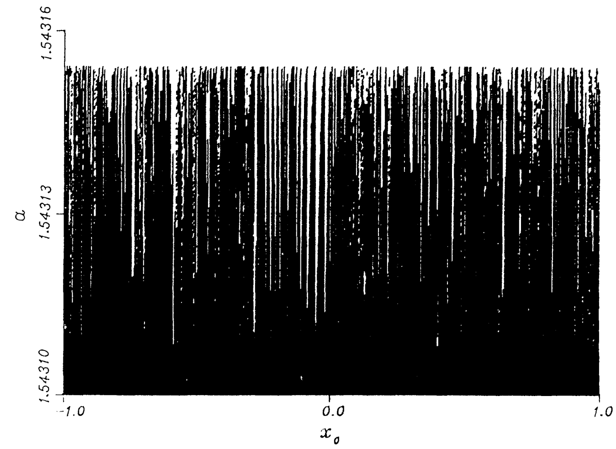

FIG. 8. $x_d = 0$: Basins of attraction for the 13- and 9-cycles over an interval of a, where the shaded regions represent 13-cycles and unshaded regions 9-cycles.

Behavior similar to the above is also found for maps with other values of x_d, for example for $x_d = 0.1$. Hence, for this map, we can understand why for some values of x_0, say 0.5, the cyclic sequence $37 \to 103 \to 37 \to 103 \to 37 \to 103 \to \cdots$ is obtained and why for other values of x_0, say 0, the length of the cycle jumps from 37 to 103 and remains at 103 until the onset of chaos.

Diagrams similar to Fig. 7 would still be obtained for bifurcations within an inverse cascade for the map F with other values of x_d, including 0. For example, consider the inverse cascade

$$\cdots \to 25 \to 21 \to 17 \to 13 \to 9$$

belonging to the map given by Eq. (1). In Fig. 8, we display the basins of attraction for the 13- and 9-cycles. A dark dot on the diagram denotes a 13-cycle, whereas a white dot indicates a 9-cycle. The lower portion of Fig. 8 is totally black, whereas the upper portion is totally white. This implies that for a value of a far away from the bifurcation parameter, only one basin of attraction exists. This property holds in general, including the period 2 to period 4 bifurcations shown in Fig. 7.

VII. OTHER FEATURES

A. Cycle elements in relation to termination of inverse cascades

Immediately following the end of the first-level inverse cascades of both the maps with $x_d = 0.01$ and 0.1, we observe from Table I that there exist cycles with periods identical to the first term of the first-level inverse cascade. Yet we do not consider these cycles as being part of the inverse cascades, even though superficially they seem to belong to the first terms of these cascades. This conclusion is based on the observation that rule I does not hold for these cycles but it holds within these two cascades.

Further, we have observed that for any cascade belonging to any map F with any value of x_d, regardless of whether rule I or rule II is satisfied, the number of positive cycle elements remains unchanged throughout the whole range of a occupied by any one term. However, in the case of the maps with $x_d=0.01$ and 0.1, the number of positive cycle elements of the cycles occurring in the range of the first terms of the inverse cascades is different from that of the cycles just after the end of the cascades. Hence, a signature for distinguishing the two consecutive ranges of cycles with identical periods, where only one of the ranges belongs to the first term of the cascade and the other lies outside the cascade, is that in going from one range to the other, the number of positive cycle elements of these cycles changes. For example, for the period 6 cycles when $x_d=0.01$, those that constitute the first term of the inverse cascade have six positive cycle elements. The other 6-cycles immediately after the end of the cascade have one negative and five positive cycle elements instead.

B. Signature for identifying inverse cascades

For the map given by Eq. (1), we find that bifurcations within a cascade occur whenever one of the cycle elements approaches the discontinuity of the map, which is $x_d=0$ [1]. We find that this property is still preserved for all the inverse cascades of the map F, thus making this a universal property.

We can use this property to confirm whether any unusual term should belong to an inverse cascade. If it does, then at the bifurcation point, one of the cycle elements should be extremely close to x_d. For example, from Table I, we can see that some terms neither belong to the inverse cascades, nor do they form part of the period-doubling sequences. For instance, for the map with $x_d=-0.01$, the consecutive terms 8, 6, and 4 do not form an inverse cascade, as during bifurcations from the 8-cycle to the 6-cycle, or from the 6-cycle to the 4-cycle, none of the cycle elements is close to $x_d=-0.01$. This classification is supported by the fact that if these terms were to be part of an inverse cascade, then we should be able to find the terms $10, 12, 14, \ldots$ and in fact, we cannot find them. Similarly, we can conclude that the consecutive terms 6 and 4 belonging to the map with $x_d=-0.1$ do not form an inverse cascade.

C. Imaginary cycles

If we ascribe to either the summation rule or its modified version in the chaotic region, then we should expect to find a certain term somewhere within the sequence belonging to the map F given by Eq. (2). However, for some values of x_d, it is impossible to find it computationally. We shall refer to these cycles as *imaginary cycles.*

Let us illustrate with some numerical examples. For the map F with $x_d=-0.1$, we observe an increasing sequence

$$28\to 44\to 60\to 76\to 92\to 108\to \cdots$$

followed by a decreasing sequence

$$\cdots \to 100\to 84\to 68\to 52\to 36\to 20\ ,$$

each with a common difference of 16. We do not refer to these sequences as direct and inverse cascades, since all the terms, except 28 and 44, lie in the chaotic region and thus are interposed with chaos. We expect the 16-cycles to occur after the end of the increasing sequence, so that we can think of the terms of the two sequences as being generated by the modified summation rule. However, we have been unable to find the 16-cycles and accordingly they are regarded as imaginary cycles.

Another example is provided by the map with $x_d=0.4$. After the 12-cycles shown in Table I, which lie in the periodic region, we observe higher-level inverse and direct cascades that seem to be generated by the summation rule between the terms 12 and 10. However, as this 10-cycle could not be found, we again refer to it as an imaginary cycle.

Though these imaginary cycles do not exist, as we cannot find them computationally, it is still convenient to think that they do exist, as we can use the summation rule or its modified version to predict the periods of real cycles.

VIII. DISCUSSION AND CONCLUSIONS

We have seen above that the discontinuous maps F given by Eq. (2), with all values of x_d including 0 but excluding -0.1, possess at least a first-level inverse cascade. Further, we have also found first-level inverse cascades in many other discontinuous maps, an example of which is given by the following equation

$$x_{n+1}=\begin{cases}1-\epsilon_1-a_1|x_n|^{z_1} & \text{if } x_n>x_d\ ,\\ 1-\epsilon_2-a_2|x_n|^{z_2} & \text{if } x_n\le x_d\ ,\end{cases} \qquad (3)$$

with different combinations of a_1, a_2, z_1, and z_2, with $\epsilon_1\neq\epsilon_2$. The amplitudes a_1 and a_2, and likewise the exponents z_1 and z_2, need not be equal. Thus we can conclude that the main requirement for the existence of first-level inverse cascades is the presence of a sectional discontinuity in the map, which can be located at many possible places and not necessarily at its extremum.

While for all values of x_d except -0.1, the map F described by Eq. (2) possesses at least a first-level inverse cascade, the number of inverse cascades is not a universal property, since for some values of x_d, there are six, and for others, only one.

We have also found that each of the first-level inverse cascades belonging to the map F with any value of x_d including 0, has a common difference equal to the period immediately below its accumulation point. Thus this rule appears to apply universally for general discontinuous maps with inverse cascades.

Bifurcations within an inverse cascade occur whenever one of the cycle elements approaches the discontinuity of the map F for all values of x_d including 0. Thus this is a universal property.

For the map with $x_d = 0$ given by Eq. (1), either rule I and the summation rule or rule II hold [1]. We have also found that for the map F given by Eq. (2) with any value of x_d, either rule I holds, in which case new cycles should exist between any two consecutive terms of the first-level inverse cascade and the periods of these new cycles can be determined from the summation rule, or rule II holds, implying that no new cycle can exist between any two consecutive terms of a first-level inverse cascade. Hence, these three rules are universal for these discontinuous maps, and we suspect that they must also hold for other types of discontinuous maps.

When the summation rule holds, there exists a rule for approximating the values of the cycle elements belonging to the new cycles lying between two first-level consecutive terms. This is given by the superimposition rule: the positions of the new cycle elements are given approximately by the superimposition of those of the cycle elements of the two consecutive terms.

The modified summation rule, for use in the chaotic region, does not have as widespread applications as the summation rule in the periodic regions. There are some values of x_d where the modified summation rule holds and some others where it does not, implying that this rule is not universal.

For the maps F given by Eq. (2) with values of x_d where the modified summation rule does not hold in the chaotic regions, we observe the following period-doubling sequences $2 \to 4 \to 8 \to 16 \to 32 \to \cdots$ either in the periodic or in the chaotic regions. Further, for each of these maps, period doublings occur at the same values of a_{PD}.

We find that the modified summation rule and the above-mentioned period-doubling sequence are mutually exclusive in the sense that if this particular period-doubling sequence exists, then the modified summation rule does not hold, and if the modified summation rule holds in the chaotic region, then this particular period-doubling sequence cannot exist though other types of period-doubling sequences may exist.

We find that there exists one, but sometimes two, basins of attraction for the map at different values of x_d. Further, when the value of a is near any bifurcation parameter, there exist two basins, and for other values of a, there is only one.

Based on our definition of the inverse cascade, we do not agree with Sousa Vieira and co-workers that chaos first occurs at the "accumulation point of the accumulation points" for the map given by Eq. (1) [3–6]. Further, for the map F with $x_d = -0.01$, 0.6, and 0.65, chaos obviously cannot occur by this route, as for each of these maps there exists only one accumulation point.

For the map F with x_d lying between 0.2 and 0.3, the route to chaos is definitely by period doublings, with the values of a_{PD} different from those of the logistic map.

Further, for the map F with $x_d = 0.1$, we find that the route to chaos is by the "alternating" route when x_0 is chosen to be 0.5. The character of the bifurcation involved in this alternating route is quite different from that of the *period-doubling* route.

For the same value of $x_d = 0.1$, the route to chaos is different for this map F when x_0 is chosen to be 0 instead. The route is then clearly basin-dependent. For this and the maps with values of x_d equal to -0.3, -0.2, -0.1, -0.01, 0.01, 0.4, 0.5, 0.6, 0.65, 0.7, or 0.8, and $x_0 = 0.5$, chaos occurs either immediately after the end of the cycles given in Table I or just after some terms that appear to belong to higher-level inverse and direct cascades. The nature of this route to chaos for these maps is still unknown.

It is clear that a universal route to chaos for the maps F given by Eq. (2) with different values of x_d does not exist, since the route can be by period doubling, or by an alternating sequence, or by a process that is still unknown.

ACKNOWLEDGMENT

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APPENDIX

The following rules and conjectures are taken from Ref. [1].

1. Rule I: Existence of cycles between any two consecutive Δ^1_{jk}'s

There will exist new cycles between any two consecutive Δ^1_{jk}'s of a first-level inverse cascade if there is an integer n denoted by n_c such that, for $n \geq n_c$, the converging iterates to any cycle element in either one of these two ranges assume either the form

$$x_n > x_{n+P} > x_{n+2P} \cdots > x_{n+(m-1)P} > x_{n+mP} > x_{n+(m+1)P} \cdots ,$$

or the form

$$x_n < x_{n+P} < x_{n+2P} \cdots < x_{n+(m-1)P} < x_{n+mP} < x_{n+(m+1)P} \cdots ,$$

where the general index $n+mP$ is an integer, x_n the nth iterate, and P the period of cycles in either one of the two consecutive ranges.

2. Conjecture I

If there exist new cycles between any two consecutive terms of a first-level inverse cascade, then new cycles will also exist between any other two consecutive terms of the same cascade.

3. Rule II: Nonexistence of cycles between any two consecutive Δ^1_{jk}'s

No new cycles will exist between any two consecutive Δ^1_{jk}'s of a first-level inverse cascade if there is an integer n_d such that, for $n \leq n_d$, the converging iterates to any cycle element in either one of these two ranges assume either the form

$$x_n > x_{n+P}, \quad x_{n+P} < x_{n+2P}, \quad x_{n+2P} > x_{n+3P}, \ldots, \quad x_{n+2mP} > x_{n+(2m+1)P}, \quad x_{n(2m+1)+P} < x_{n+(2m+1)P}, \ldots,$$

with

$$x_n > x_{n+2P} > x_{n+4P} \cdots > x_{n+(2m-2)P} > x_{n+2mP} > x_{n+(2m+2)P} \cdots$$

and

$$x_{n+P} < x_{n+3P} < x_{n+5P} \cdots < x_{n+(2m-1)P} < x_{n+(2m+1)P} < x_{n+(2m+3)P} \cdots,$$

or the form

$$x_n < x_{n+P}, \quad x_{n+P} > x_{n+2P}, \quad x_{n+2P} < x_{n+3P}, \ldots,$$

$$x_{n+2mP} < x_{n+(2m+1)P}, \quad x_{n+(2m+1)P} > x_{n+(2m+2)P}, \ldots,$$

with

$$x_n < x_{n+2P} < x_{n+4P} \cdots < x_{n+(2m-2)P} < x_{n+2mP} < x_{n+(2m+2)P} \cdots$$

and

$$x_{n+P} > x_{n+3P} > x_{n+5P} \cdots > x_{n+(2m-1)P} > x_{n+(2m+1)P} > x_{n+(2m+3)P} \cdots,$$

where the general index $n+mP$ is an integer, x_n the nth iterate, and P the period of cycles in either one of the two consecutive ranges.

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